Int. J. Multiphase Flow Vol. 9, No. 5, pp. 561-574, 1983
Printed in Great Britain.

0301-9322/83 \$3.00 + .00
Pergamon/Elsevier

BOUNDARY CONDITIONS FOR DARCY'S FLOW THROUGH POROUS MEDIA

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(*Received 4 July 1982; in revised form 15 January 1983*)

Abstract—A novel formulation for the boundary conditions to be applied at a porous surface is proposed. Interfaces between porous and clear media and porous and solid media are considered. The well known Beavers & Joseph boundary condition is applicable for interfaces between porous and clear media. An equivalent boundary condition is obtained for interfaces between porous and impermeable media, namely, $\mathbf{v}\cdot\mathbf{n} = \sqrt{(k)}\nabla_t \cdot \mathbf{v}_t$ where $\mathbf{v}$ is the velocity field inside the porous medium, $\mathbf{n}$ denotes a unit vector normal to the interface pointing towards the porous medium, k stands for the permeability and the subscript t refers to components tangential to the interface. A sample problem is solved for the flow fields exterior to a porous spherical particle and interior to it, assuming that the particle has a rigid concentric spherical core and that the submerging flow field is Newtonian, Stokesian and uniform at infinity. Both Brinkman's equation and Darcy's law are utilized to obtain general forms of the velocity and pressure fields. Comparison of the two solutions yields the desired boundary conditions applicable to the Darcy problem.

1. INTRODUCTION

In the past years various aspects of the creeping Newtonian flow through an isotropic porous medium were investigated. Theoretical studies (Slattery 1969; Saffman 1971; Lundgren 1972) have shown that the most satisfactory solution of the problem is that based upon Brinkman's extension of Darcy's law.

Moreover, Brinkman's equation is compatible with the existence of a boundary layer region at the edge of the porous medium, which has been indicated by the experiments of Beavers & Joseph (1967). However, solving Brinkman's equation is harder by far than solving Darcy's equation, and a closed analytical solution can be found only for very simple geometries. Consequently, a comparison between the results obtained applying the two equations is of great importance, in order to determine the boundary conditions that allow the Darcy solution to best approximate the solution derived from Brinkman's equation.

Darcy's law, $\nabla p = -(\mu/k)\,\mathbf{v}$, can be viewed as a lower order approximation in k of Brinkman's equation, $\nabla p = -(\mu/k)\,\mathbf{v} + \mu\nabla^2\mathbf{v}$, as demonstrated in the following.

Define the non-dimensional flow parameters

$$\hat{p} = p/(\mu Ua/k)\ ;\quad \hat{\mathbf{v}} = \mathbf{v}/U\ ;\quad \hat{\nabla} = a\nabla$$

where U stands for a characteristic velocity inside the porous field, a denotes a characteristic dimension of the field and $\mu Ua/k$ is the dimensional pressure derived from the differential equation in order to preserve the pressure term. Thus, Brinkman's equation possesses the non-dimensional form

$$\hat{\nabla}\hat{p} = -\hat{\mathbf{v}} + \epsilon^2\hat{\nabla}^2\hat{\mathbf{v}}$$

where

$$\epsilon = \frac{\sqrt{k}}{a}.$$

A regular power expansion in ϵ for $\hat{p}$ and $\hat{v}$ yields:

$$\hat{p} = p_0 + \epsilon p_1 + \epsilon^2 p_2 + 0(\epsilon^3)$$

$$\hat{\mathbf{v}} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + 0(\epsilon^3)$$

where the existence of odd powers in ϵ is *a posteriori* explained.

Introduction of the foregoing series into Brinkman's equation and equating common powers of ϵ results in the following differential equations:

$$\hat{\nabla} p_0 = -\mathbf{v}_0$$

$$\hat{\nabla} p_1 = -\mathbf{v}_1$$

$$\hat{\nabla} p_2 = -\mathbf{v}_2 + \hat{\nabla}^2 \mathbf{v}_0.$$

Thus, solutions can be derived for the zeroth and first order fields, $\mathbf{v}_0$, p_0, $\mathbf{v}_1$, p_1 when applying Darcy's law. Brinkman's equation is needed to obtain a solution for the higher order approximation fields $\mathbf{v}_2$, p_2. Derivation of even higher order fields is feasible provided a higher order differential equation in k is known.

A striking fact is that Darcy's law provides a solution correct to order ϵ and not just for zeroth order fields p_0, $\mathbf{v}_0$. However, the elimination of the term $\mu\nabla^2\mathbf{v}$ in Darcy's law may cause difficulties normally encountered when high order derivatives are discarded. The low order differential equation cannot entertain the full set of boundary conditions and a boundary layer region is created at the edge of the porous medium. For this boundary layer region, different length scales must be chosen in order to preserve the second order derivatives, namely:

$$\bar{x} = x/a \qquad \bar{v}_x = v_x/U$$

$$\bar{y} = y/a \qquad \bar{v}_y = v_y/U \quad \bar{p} = p\Big/\left(\frac{\mu U a}{k}\right)$$

$$\bar{z} = z/\sqrt{k} \qquad \bar{v}_z = v_z\Big/\left(U\frac{\sqrt{k}}{a}\right)$$

where x, y, z are orthogonal Cartesian coordinates placed on the boundary, z being normal to the layer and v_x, v_y, v_z are the corresponding velocities. A stretched coordinate $\bar{z}$ was defined to focus on the rapid changes through the boundary layer and the scaling of $\bar{v}_z$ was derived from the continuity equation. Thus, the governing field equations in the boundary layer region are:

$$\frac{\partial \bar{p}}{\partial \bar{x}} = -\bar{v}_x + \frac{\partial^2 \bar{v}_x}{\partial \bar{z}^2} + 0(\epsilon^2)$$

$$\frac{\partial \bar{p}}{\partial \bar{y}} = -\bar{v}_y + \frac{\partial^2 \bar{v}_y}{\partial \bar{z}^2} + 0(\epsilon^2)$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = -\epsilon^2\left(\bar{v}_z - \frac{\partial^2 \bar{v}_z}{\partial \bar{z}^2}\right) + 0(\epsilon^4)$$

which manifest the very important fact that pressure variations through the boundary layer are of order ϵ^2 and can be neglected when solution of order ϵ is desired. The scaling of z by $\sqrt{k}$ preserves second order derivatives and leads to the observation that the thickness of the boundary layer is expected to be of order $\sqrt{k}$. Hence a power expansion in $\epsilon = \sqrt{(k)}/a$ (rather than ϵ^2) for the pressure and velocity was assumed. It seems, therefore, that a consistent approximate solution of the flow inside a porous medium can be derived only when the outer (Darcy) regular solution is matched with the boundary layer solution. However, if we desire to use only Darcy's law because of its relative simplicity and wide domain of existance, we have to modify the boundary conditions in such a manner that the Darcy solution satisfies its *asymptotic* value on the boundary rather than the physical value satisfied by the boundary layer. In such a case it is *a priori* evident that the solution near the surface (at distances of order $\sqrt{k}$) is incorrect and no solution of higher order than ϵ for p and $\mathbf{v}$ have any meaning. Consequently, the *artificial* boundary conditions for the Darcy problem must possess zeroth *and first* order terms in ϵ to be compatible with the order of solution provided by Darcy's law.

In a previous work, Neale *et al.* (1973) have produced a review of the different boundary conditions that can be applied to the Darcy law. They concluded that the most satisfactory boundary condition to be considered is the one presented by Saffman (1971) for interfaces between clear fluid and porous medium. However, meaningful solutions cannot be achieved without introducing empirical parameters. No real effort was made to explore the correct boundary conditions to be applied at interfaces between porous and solid media.

In the present paper these statements will be discussed, comparing Brinkman's exact solution and Darcy's asymptotic solution for a sample problem, where the viscous flow field relative to an isolated permeable sphere with an impermeable core is analyzed.

The reason for chosing this particular sample problem is two fold: First, it includes two different interfaces of interest, namely, an interface between porous and clear media at the external surface of the sphere and an interface between porous and impermeable media at the core's boundary. Second, it is of significance in the processing of solid particles in a chemical reactor such as new-catalytic solid–gas reaction: particularly limestone or dolomite sulphuring, and oil shale combustion.

2. MODEL DESCRIPTION

The viscous flow past a spherical particle is studied, assuming that the particle is described by an unreacted core model. According to this model, two distinct regions inside the particle are considered: the solid product porous layer and the unreacted impermeable core.

In order to study the behaviour of the system, we choose to impose a uniform negative pressure gradient in the $+z$ direction, thus introducing a uniform flow field $\mathbf{U}$ defined by $\mathbf{U} = (0, 0, -V)$ in Cartesian coordinates (x, y, z). The external flow is governed by the Stokes and the continuity equations. Thus, for an incompressible Newtonian fluid we may write:

$$\mu\nabla^2\mathbf{v} = \nabla p \qquad [1]$$

$$\text{for } r > b$$

$$\nabla\cdot\mathbf{v} = 0 \qquad [2]$$

where $\mathbf{v}$ and p denote the velocity and pressure fields, respectively, μ stands for the viscosity of the fluid and b denotes the outer radius of the particle.

Within the porous mass, the equation of motion most widely adopted to describe a creeping Newtonian flow through an isotropic porous medium is the empirical Darcy law:

$$-\frac{\mu}{k}\hat{\mathbf{v}} = \nabla\bar{p} \qquad a < r < b \qquad [3]$$

where k is the permeability of the porous medium and a denotes the radius of the impermeable core.

The incompressibility condition is:

$$\nabla\cdot\hat{\mathbf{v}} = 0 \qquad [4]$$

where "$\hat{}$" denotes an ensemble average quantity within the porous medium, and "$\bar{}$" denotes an interstitial-average quantity within the porous medium.

However, theoretical (Slattery 1963; Saffman 1971; Lundgren 1972) and experimental (Beavers & Joseph 1967) studies have led to the conclusion that Brinkman's extension of the Darcy equation is preferable to the original Darcy law in its simple form, i.e.

$$-\frac{\mu}{k}\hat{\mathbf{v}}+\tilde{\mu}\nabla^2\hat{\mathbf{v}}=\nabla\bar{p} \qquad [5]$$

where $\tilde{\mu}$ denotes an effective viscosity, which is identified with μ for high-porosity permeable particle (Lundgren 1972). Howells (1974) and Hinch (1977) confirm the validity of [5] with $\tilde{\mu}=\mu$, and $\bar{p}\rightarrow p$ (for large void fractions), by considering slow flow in random arrays of fixed spheres and for suspensions, respectively. We henceforth rely upon the fact that Brinkman's equation is an acceptable governing field equation for the entire domain of the porous medium. This can be done since it can entertain the full set of boundary conditions, although no rigorous proof exists of its validity close to the boundaries.

3. EXACT SOLUTION OF THE PROBLEM APPLYING BRINKMAN'S EQUATION

In view of the axisymmetric nature of the problem, the azimuthal coordinate ϕ may be suppressed. The boundary conditions which must be satisfied may be expressed as follows:

the asymptotic conditions:

$$\lim_{r\to\infty}\mathbf{v}(r,\theta)=-V\mathbf{i}_z \qquad [6]$$

$$\lim_{r\to\infty}p(r,\theta)=p_\infty \qquad [7]$$

the no-slip condition on the rigid core $r=a$:

$$\hat{\mathbf{v}}(a,\theta)=0 \qquad [8]$$

the continuity of the velocity field at the interface $r=b$:

$$\mathbf{v}(b,\theta)=\hat{\mathbf{v}}(b,\theta) \qquad [9]$$

the continuity of the stress vector normal to the interface $r=b$:

$$\tau_{r\theta}(b,\theta)=\hat{\tau}_{r\theta}(b,\theta) \qquad [10]'$$

$$\tau_{rr}(b,\theta)=\hat{\tau}_{rr}(b,\theta). \qquad [11]'$$

Note that, since:

$$\tau_{r\theta}=\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right),$$

from the continuity of $\mathbf{v}$ at the interface $r=b$, the boundary condition [10]′ becomes:

$$\frac{\partial v_\theta}{\partial r}(b,\theta)=\frac{\partial\hat{v}_\theta}{\partial r}(b,\theta). \qquad [10]$$

Moreover, from the continuity of $\mathbf{v}$ at the interface $r=b$ and from the continuity equations [2] and [4] we obtain:

$$\frac{\partial v_r}{\partial r}(b, \theta) = \frac{\partial \hat{v}_r}{\partial r}(b, \theta).$$

Consequently, since by definition

$$\tau_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

the boundary condition [11]′ can be written more simply in the following way:

$$p(b, \theta) = \hat{p}(b, \theta). \qquad [11]$$

Now we seek solutions of the Stokes equations [1] and [2] for the outer flow and of the Brinkman equations [5] and [4] for the internal flow in terms of their respective stream function ψ and $\hat{\psi}$ as follows (Happel & Brenner 1965):

$$E^4\psi = 0 \qquad r > b \qquad [12]$$

$$E^4\hat{\psi} - \frac{1}{k}E^2\hat{\psi} = 0 \qquad a < r < b \qquad [13]$$

where:

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

The general solutions of the above equations are:

$$\psi(\xi, \theta) = \frac{kV}{2}\left[\frac{A}{\xi} + B\xi + C\xi^2 + D\xi^4\right] \qquad \xi > \beta$$

$$\hat{\psi}(\xi, \theta) = \frac{kV}{2}\left[\frac{E}{\xi} + F\xi^2 + GS_1(\xi) + HS_2(\xi)\right] \qquad \alpha < \xi < \beta$$

where:

$$\xi = \frac{r}{\sqrt{k}}, \quad \alpha = \frac{a}{\sqrt{k}}, \quad \beta = \frac{b}{\sqrt{k}},$$

$$S_1(\xi) = \frac{\mathrm{Ch}\xi}{\xi} - \mathrm{Sh}\xi,$$

$$S_2(\xi) = \frac{\mathrm{Sh}\xi}{\xi} - \mathrm{Ch}\xi$$

and Sh and Ch stand for the hyperbolic sine and cosine respectively. Hence the flow field for the outer flow is given by the following expressions:

$$v_r = -V\cos\theta\left[\frac{A}{\xi^3} + \frac{B}{\xi} + 1\right] \qquad [14]$$

$$v_\theta = \frac{V}{2}\sin\theta\left[-\frac{A}{\xi^3}+\frac{B}{\xi}+2\right] \qquad [15]$$

$$p = p_\infty - \frac{\mu}{\sqrt{k}}VB\frac{\cos\theta}{\xi^2} \qquad [16]$$

where the asymptotic conditions [6] and [7] have already been fulfilled. For the inner flow we have the following general form:

$$\hat{v}_r = -V\cos\theta\left[\frac{E}{\xi^3}+F+G\frac{S_1(\xi)}{\xi^2}+H\frac{S_2(\xi)}{\xi^2}\right] \qquad [17]$$

$$\hat{v}_\theta = \frac{1}{2}V\sin\theta\left[-\frac{E}{\xi^3}+2F+G\frac{S_1'(\xi)}{\xi}+H\frac{S_2'(\xi)}{\xi}\right] \qquad [18]$$

$$\hat{p} = p_\infty - \frac{\mu}{\sqrt{k}}V\cos\theta\left[\frac{E}{2\xi^2}-F\xi\right]. \qquad [19]$$

Finally, applying the boundary conditions [8]–[11], we find that the coefficients must assume the following values:

$$A = \Delta^{-1}\frac{\beta^3}{3}\left\{\frac{1}{\beta}-\frac{3}{2\beta^2}(\beta^2+2)(\gamma_2\delta_1-\gamma_1\delta_2)-3\frac{\alpha}{\beta^3}\delta_1-\frac{\alpha^2+3}{\beta^3}\delta_2\right.$$

$$\left.+\gamma_1\mathrm{Ch}(\beta-\alpha)-\gamma_2\mathrm{Sh}(\beta-\alpha)\right\} \qquad [20]$$

$$B = \Delta^{-1}\frac{\beta}{2}(\gamma_2\delta_1-\gamma_1\delta_2) \qquad [21]$$

$$E = \frac{\Delta^{-1}}{3}\{2\beta^2-3\alpha\delta_1-(\alpha^2+3)\delta_2\} \qquad [22]$$

$$F = \frac{\Delta^{-1}}{3\alpha\beta}(\alpha+\beta\delta_2) \qquad [23]$$

$$G = \Delta^{-1}\{\mathrm{Sh}\alpha(\delta_1-\gamma_1)+\mathrm{Ch}\alpha(\delta_2-\gamma_2)\} \qquad [24]$$

$$H = -\Delta^{-1}\{\mathrm{Ch}\alpha(\delta_1-\gamma_1)+\mathrm{Sh}\alpha(\delta_2-\gamma_2)\} \qquad [25]$$

where:

$$\Delta = (9\alpha\beta)^{-1}\{6\alpha-3(\alpha^2-1)\mathrm{Sh}(\beta-\alpha)-(2\beta^3+\alpha^3+3\alpha+3\beta)\mathrm{Ch}(\beta-\alpha)\}$$

$$\delta_1 = \frac{\mathrm{Ch}(\beta-\alpha)}{\beta}-\mathrm{Sh}(\beta-\alpha)$$

$$\delta_2 = \frac{\mathrm{Ch}(\beta-\alpha)}{\beta}-\mathrm{Ch}(\beta-\alpha)$$

$$\gamma_1 = \frac{2\beta^3 + \alpha^3 + 3\alpha}{3\alpha\beta}$$

$$\gamma_2 = -\frac{\alpha}{\beta}.$$

The drag force exerted by the flow on the sphere is given by Happel & Brenner (1965):

$$F_D = 8\pi\mu \lim_{r\to\infty} \frac{(\psi - \psi_\infty)}{r \sin^2\theta}$$

that is:

$$F_D = 4\pi\mu V\sqrt{(k)}B.$$

Consequently, defining the Ω coefficient as the ratio of the resistance experienced by the permeable sphere to that experienced by an impermeable sphere with the same dimensions, we obtain:

$$\Omega = \frac{1}{\beta}\frac{\mathrm{Ch}(\beta-\alpha)[2\beta^4 + \alpha^3\beta - 3\alpha^2 + 3\alpha\beta] - \mathrm{Sh}(\beta-\alpha)[2\beta^3 + \alpha^3 - 3\alpha^2\beta + 3\alpha]}{\mathrm{Ch}(\beta-\alpha)[2\beta^3 + \alpha^3 + 3\alpha + 3\beta] + \mathrm{Sh}(\beta-\alpha)[3\alpha^2 - 3] - 6\alpha}. \qquad [26]$$

As a particular case, consider a coreless porous sphere, for which we obtain:

$$\Omega = \frac{2\beta^2[1 - (\mathrm{Th}\beta/\beta)]}{2\beta^2 + 3[1 - (\mathrm{Th}\beta/\beta)]}$$

This result coincides with the one mentioned in Neale *et al.* (1973).

From figure 1 it appears that the Ω-curves corresponding to different values of a/b converge rapidly towards a common value (for $\beta > 20$, the difference in Ω between the case of $a/b = 0$ and that of $a/b = 0.9$ is less than 0.4%) and that, more slowly, they all converge to unity.

In fact, as it will be shown later, for large values of the non-dimensional radius, the outer flow at a zeroth order approximation is equal to that past an impermeable sphere, and up to a first order approximation it is not influenced by the presence of the core. In the following part of the paper we will examine in detail the foregoing results, assuming that the non-dimensional radii α and β are large compared to unity.

4. THE ASYMPTOTIC EXPANSION OF THE INNER AND OUTER FLOWS—(NO CORE CASE)

For large values of β (i.e. the mean diameter of the pores is much smaller than the diameter of the particle), the inner velocity and pressure fields [17]–[19] can be expanded in terms of $\epsilon = (1/\beta) = (\sqrt{k}/b)$ as follows:

$$\hat{v}_r = -V\cos\theta F\left[1 + \zeta_0^r(r)\,\mathrm{e}^{-(b-r)/\sqrt{k}} + o\left(\mathrm{e}^{-b/\sqrt{k}}\right)\right] \qquad [27]$$

$$\hat{v}_\theta = V\sin\theta F\left[1 + \zeta_0^\theta(r)\,\mathrm{e}^{-(b-r)/\sqrt{k}} + o\left(\mathrm{e}^{-b/\sqrt{k}}\right)\right] \qquad [28]$$

$$\hat{p} = p_\infty + \frac{1}{\epsilon^2}\mu V\cos\theta F\frac{r}{b^2} \qquad [29]$$

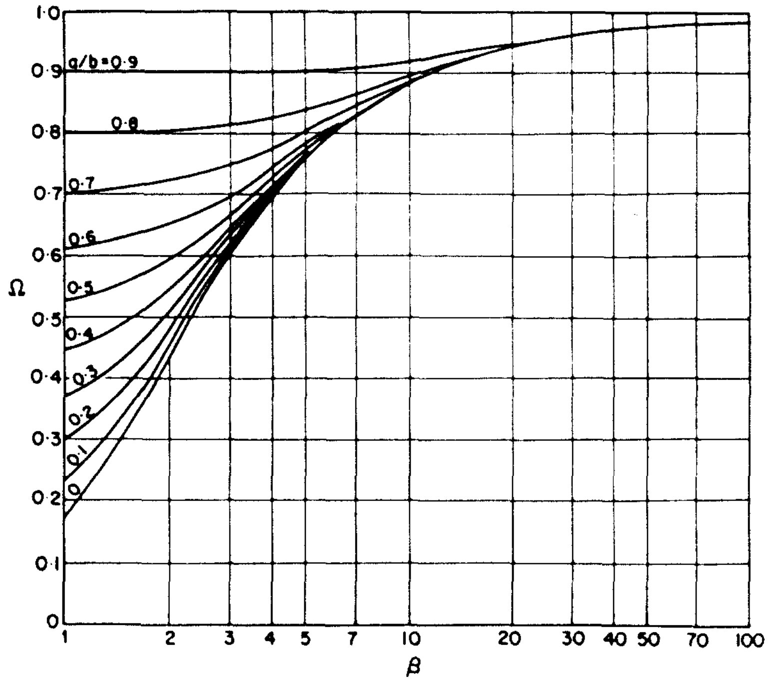

Figure 1. The ratio Ω of the resistance experienced by a permeable sphere to that exerted on an impermeable sphere of the same dimension vs β.

where:

$$F = \frac{3}{2}\epsilon^2\left(1 - \epsilon - \frac{3}{2}\epsilon^2\right) + O(\epsilon^5)$$

$$\zeta_0{}^r = 2\frac{b^2}{r^3}\frac{r - \epsilon b}{1 - \epsilon}$$

$$\zeta_0{}^\theta = \frac{b(r^2 - \epsilon r b + \epsilon^2 b^2)}{\epsilon r^3(1 - \epsilon)}.$$

This result manifests the existence of a boundary layer of thickness $\sqrt{k}$ at the interface, in which both the velocities $\hat{v}_r$ and $\hat{v}_\theta$ increase up to their interface values. However, the rate of change of $\hat{v}_r$ and $\hat{v}_\theta$ vary across the boundary layer: at the interface $\hat{v}_\theta$ is an order of magnitude (that is $(1/\epsilon)$ times) larger than $\hat{v}_r$, while far from the boundary layer they have the same order of magnitude. Finally note that the pressure is not influenced by the presence of the boundary layer, that is there is no pressure boundary layer.

Analogously, the outer flow [14]–[16] can be developed into powers of ϵ, obtaining:

$$v_r = -V\cos\theta\left\{\left(\frac{1}{2}\frac{b^3}{r^3} - \frac{3}{2}\frac{b}{r} + 1\right) + \epsilon\left(-\frac{3}{2}\frac{b^3}{r^3} + \frac{3}{2}\frac{b}{r}\right) + \epsilon^2\left(\frac{9}{4}\frac{b^3}{r^3} + \frac{9}{4}\frac{b}{r}\right) + O(\epsilon^3)\right\} \quad [30]$$

$$v_\theta = \frac{V}{2}\sin\theta\left\{\left(-\frac{1}{2}\frac{b^3}{r^3} - \frac{3}{2}\frac{b}{r} + 2\right) + \epsilon\left(\frac{3}{2}\frac{b^3}{r^3} + \frac{3}{2}\frac{b}{r}\right) + \epsilon^2\left(-\frac{9}{4}\frac{b^3}{r^3} + \frac{9}{4}\frac{b}{r}\right) + O(\epsilon^3)\right\} \quad [31]$$

$$p = p_\infty + \frac{3}{2}\mu V\cos\theta \frac{b}{r^2}\left[1 - \epsilon - \frac{3}{2}\epsilon^2\right) + O(\epsilon^3)\Big]. \quad [32]$$

Thus, in a zeroth order approximation, the flow field is identical to that past an impermeable sphere.

At the interface, the first meaningful term for v_r is the third, while for v_θ it is the second. This confirms what has already been shown, that at the interface v_θ is an order of magnitude larger than v_r. Moreover, it is easy to verify that as a first order approximation in ϵ the following relation holds:

$$v_\theta(b,\theta) = \sqrt{k}\,\frac{\partial v_\theta(b,\theta)}{\partial r}. \quad [33]$$

This is Saffman's condition, when Saffman's λ-parameter is equal to one. It may be of some significant to note here that much of the data of Beavers & Joseph (1967) can be reasonably correlated by assigning λ the value of unity.

At the same approximation of order ϵ we have:

$$v_r(b,\theta) = 0. \quad [34]$$

Substituting the above boundary conditions [33] and [34] into the general solution for the outer flow [14]–[16], the complete "Brinkman" solution [30]–[32] is fitted up to the first order term of the expansion into powers of ϵ.

It should be emphasized that the boundary conditions [33] and [34] refer only to the outer flow, which consequently can be calculated (up to the first order approximation in ϵ) without imposing any coupling condition with the inner flow at the interface.

A physical explanation of the boundary condition [33] is based on the existence of a boundary layer in the inner part of the interface; since the tangential velocity across the boundary layer drops from its interface value to a value which is $(1/\epsilon)$ times smaller, we may write, qualitatively:

$$0 \cong \hat{v}_\theta(b - \sqrt{k}) = \hat{v}_\theta(b) - \sqrt{k}\left.\frac{\partial \hat{v}_\theta}{\partial r}\right|_{r=b} + O(\epsilon^2).$$

Owing to the continuity of the tangential velocity and its derivative across the interface (see [9] and [10]), this relation holds for v_θ and $\partial v_\theta/\partial r$ of the outer field as well, and Saffman's result follows. Note that the profile of the radial velocity of the boundary layer is not steep enough to apply the foregoing result for v_r as well.

5. THE ASYMPTOTIC EXPANSION OF THE INNER AND OUTER FLOWS FOR A PARTICLE POSSESSING A CORE

Let us discuss the complete solution, again assuming that the non-dimensional radius ξ is very large, and expanding [14]–[16] into powers of $\epsilon = \sqrt{(k)}/b$ (a and b fixed). A rather elaborate expansion of the coefficients A and B of the general solution, [25] and [26], yields for the outer flow:

$$v_r = -V\cos\theta\left\{\left(\frac{1}{2}\frac{b^3}{r^3} - \frac{3}{2}\frac{b}{r} + 1\right) + \epsilon\left(-\frac{3}{2}\frac{b^3}{r^3} + \frac{3}{2}\frac{b}{r}\right)\right.$$

$$\left. + \epsilon^2\frac{9}{2}\frac{b^3}{(2b^3 + a^3)}\left(\frac{b^3}{r^3} + \frac{b}{r}\right) + O(\epsilon^3)\right\} \quad [35]$$

$$v_\theta = \frac{1}{2} V \sin\theta \left\{ \left(-\frac{1}{2}\frac{b^3}{r^3} - \frac{3}{2}\frac{b}{r} + 2 \right) + \epsilon \left(\frac{3}{2}\frac{b^3}{r^3} + \frac{3}{2}\frac{b}{r} \right) + \epsilon^2 \frac{9}{2}\frac{b^3}{(2b^3+a^3)} \left(-\frac{b^3}{r^3} + \frac{b}{r} \right) + 0(\epsilon^3) \right\} \qquad [36]$$

$$p = p_\infty + \frac{3}{2}\mu V \cos\theta \frac{b}{r^2} \left[1 - \epsilon - 3\epsilon^2 \frac{b^3}{(2b^3+a^3)} + 0(\epsilon^3) \right]. \qquad [37]$$

Comparing these expressions with the analogous results for the no-core case, we conclude that at a first order approximation the outer flow is not influenced by the presence of the core.

Consequently, the boundary conditions [33] and [34], when applied to the general solution for the outer flow [14]–[16], yields a first order approximation of the complete result. In the same way, after a rather long expansion we obtain the following expressions for the inner flow:

$$\hat{v}_r = -3V\cos\theta \frac{\epsilon^2}{Q} \left[q^r(r) + \zeta_a^r(r)\, e^{-(r-a)/\sqrt{k}} + \zeta_b^r(r)\, e^{-(b-r)/\sqrt{k}} + o(e^{-(b-a)/\sqrt{k}}) \right]$$

$$\hat{v}_\theta = \frac{3}{2} V \sin\theta \frac{\epsilon^2}{Q} \left[q^\theta(r) + \zeta_a^\theta(r)\, e^{-(r-a)/\sqrt{k}} + \zeta_b^\theta(r)\, e^{-(b-r)/\sqrt{k}} + o(e^{-(b-a)/\sqrt{k}}) \right] \qquad [38]$$

$$\hat{p} = p_\infty + \frac{3}{2}\mu V \cos\theta \frac{r}{b^2 Q} q^\theta(r)$$

where:

$$Q = (2b^3 + a^3) + \epsilon 3a^2 b + \epsilon^2 3b^2(a+b) - \epsilon^3 3b^3$$

$$q^r(r) = \frac{b^3}{r^3}(1-\epsilon)[r^3 - a(a^2 + \epsilon 3ab + \epsilon^2 3b^2)]$$

$$q^\theta(r) = \frac{b^3}{r^3}(1-\epsilon)[2r^3 + a(a^2 + \epsilon 3ab + \epsilon^2 3b^2)]$$

$$\zeta_a^r(r) = \epsilon 3 \frac{ab^4}{r^3}(1-\epsilon)(r + \epsilon b)$$

$$\zeta_a^\theta(r) = -3\frac{ab^3}{r^3}(1-\epsilon)(r^2 + \epsilon rb + \epsilon^2 b^2)$$

$$\zeta_b^r(r) = \frac{b^2}{r^3}[(2b^3 + a^3) + \epsilon 3a^2 b + \epsilon^2 3ab^2](r - \epsilon b)$$

$$\zeta_b^\theta(r) = \frac{1}{\epsilon}\frac{b}{r^3}(r^2 - \epsilon rb + \epsilon^2 b^2)[(2b^3 + a^3) + \epsilon 3a^2 b + \epsilon^2 3ab^2].$$

The expressions for the velocities are composed of three parts. The first ones, $q^r(r)$ and $q^\theta(r)$ are regular functions of r and as we will see, they are fully determined through the first term of

Brinkman's equation, namely, Darcy's law; the second and third parts manifest the existence of two boundary layers, one at the interface $r = b$ and the other at the surface of the core $r = a$. As in the coreless case, there is no boundary layer for the pressure.

Expanding the solution of Brinkman's equation outside the boundary layers we obtain:

$$\hat{v}_r = -V\epsilon^2 \cos\theta \frac{3b^3}{2b^3 + a^3}\left\{\left(1 - \frac{a^3}{r^3}\right) + \epsilon\left[-3\frac{a^2 b}{r^3} + \left(\frac{a^3}{r^3} - 1\right)\left(\frac{3a^2 b}{2b^3 + a^3} + 1\right)\right] + 0(\epsilon^2)\right\} \quad [39]$$

$$\hat{v}_\theta = \frac{1}{2} V\epsilon^2 \sin\theta \frac{3b^3}{2b^3 + a^3}\left\{\left(2 + \frac{a^3}{r^3}\right) + \epsilon\left[3\frac{a^2 b}{r^3} - \left(\frac{a^3}{r^3} + 2\right)\left(\frac{3a^2 b}{2b^3 + a^3} + 1\right)\right] + 0(\epsilon^2)\right\} \quad [40]$$

$$\hat{p} = p_\infty + \mu V \cos\theta \frac{3b^2}{2b^3 + a^3}\left\{\left(\frac{a^3}{2br^2} + \frac{r}{b}\right) + \epsilon\left[\frac{3}{2}\frac{a^2}{r^2} - \left(\frac{3a^2 b}{2b^3 + a^3} + 1\right)\left(\frac{a^3}{2br^2} + \frac{r}{b}\right)\right] + 0(\epsilon^2)\right\}. \quad [41]$$

Since V is the characteristic velocity of the outer field, $U = V\epsilon^2$ is the characteristic velocity inside the porous medium. Inspection of [39] indicates that the asymptotic value for $\hat{v}_r$ at $r = a$ does not vanish, rather the velocity component $(\hat{v}_r/U)$ is of order ϵ. Although [39]–[41] are invalid in the region $a \le r \le a + \epsilon$, an asymptotic evaluation leads to the following results:

$$\hat{v}_r(a)/U = 0(\epsilon^1)$$

$$\hat{v}_r'(a)/U = 0(\epsilon^0)$$

where the prime stands for the derivative with respect to r. The fact that a term of order ϵ^0 in $(\hat{v}_r(a)/U)$ vanishes proves to be very useful and fully agrees with the previous analysis presented in the introductory chapter ($\hat{v}_r$ replaces $\hat{v}_z$ as the normal component to the interface).

Indeed, it makes it possible to obtain a closed form expression for the boundary condition at an interface between porous and impermeable media, namely:

$$\hat{v}_r(a, \theta) = -\sqrt{(k)}\frac{\partial \hat{v}_r}{\partial r}(a, \theta) \quad [42]$$

where [39] is satisfied to the first order in ϵ. This novel boundary condition must be applied instead of the one commonly used,

$$\hat{v}_r(a, \theta) = 0 \quad [43]$$

since it is compatible with Darcy's law, as shown in the introductory chapter, and shall provide a solution for $\hat{p}$ and $\hat{\mathbf{v}}$ correct to first order in ϵ. The foregoing statement is illustrated in the next chapter.

6. SOLUTION OF THE PROBLEM USING THE DARCY LAW

As we have seen, the major contribution of the shear term $\mu\nabla^2\mathbf{v}$ in Brinkman's equation is to account for the experimentally-observed boundary zones near the edges of the porous medium. (Beavers & Joseph 1967 observed it for a porous-clear interface.)

The thickness of the boundary layer is actually quite small, of the order of $\sqrt{k}$, and outside it the contribution of the shear term in Brinkman's equation effectively reduces to Darcy's law within the largest portion of the porous medium, outside the foregoing boundary layers.

Applying Darcy's equations [3] and [4], and assuming the continuity of pressure at the interface, $r = b$, the general solution for the inner flow is the following:

$$\hat{v}_r = -V\cos\theta\left[\frac{E'}{\xi^3}+F'\right] \quad [44]$$

$$\hat{v}_\theta = \frac{1}{2}V\sin\theta\left[-\frac{E'}{\xi^3}+2F'\right] \quad [45]$$

$$\hat{p} = p_\infty - \frac{\mu}{\sqrt{k}}V\cos\theta\left[\frac{E'}{2\xi^2}-F'\xi\right] \quad [46]$$

Comparing these expressions with the general Brinkman solution (see [17]–[19]), it is easy to notice that the pressure has the same form, while in the velocity field the second part in the Brinkman solution (i.e. the part describing the boundary layer) is missing.

In order to find the values of E' and F', two boundary conditions are needed. The former is the continuity of pressure which has already been partly fulfilled in [44]–[46]. This boundary condition can be applied to the Darcy law since there is no boundary layer for pressure and consequently Darcy's law provides an exact picture of the behavior of the pressure.

However, among the boundary conditions [6]–[11], none can be applied to Darcy's law. In fact, no restriction can be placed on the shear stress term (as in [10] and [11]) because the Darcy equation is not of a sufficiently high order to entertain restrictions on this quantity.

Moreover, the velocity field at the edge of the porous medium presents a sharp change, which is incompatible with Darcy's law. Thus, we come to the conclusion that the missing boundary condition cannot be but [42], which replaces the common boundary condition [43]. In conclusion, it is easy to verify that the results obtained with Brinkman's equation (see [39]–[41]) outside the boundary layers can be fitted to a first order approximation applying the Darcy law with the following boundary conditions:

$$p(b,\theta) = \hat{p}(b,\theta)$$

$$\hat{v}_r(a,\theta) = -\sqrt{(k)}\,\frac{\partial \hat{v}}{\partial r}(a,\theta)$$

As we have already seen, the outer flow can also be fitted to a first order approximation by applying Stokes' equation with the following boundary conditions:

$$\lim_{r\to\infty} \mathbf{v}(r,\theta) = -V\mathbf{i},$$

$$\lim_{r\to\infty} p(r,\theta) = p_\infty$$

$$v_r(b,\theta) = 0$$

$$v_\theta(b,\theta) = \sqrt{(k)}\,\frac{\partial v_\theta}{\partial r}(b,\theta).$$

Note that the only coupling condition between the outer and the inner flow is the continuity of pressure, as one should expect from physical considerations.

Finally, let us rewrite the boundary conditions in a Galilean-invariant form. At an interface between clear and permeable media we have:

$$p = \hat{p} \quad [50]$$

$$(\mathbf{v}-\mathbf{v}_{\text{int}})\cdot(\underline{\mathbf{I}}-\mathbf{nn})=\sqrt{(k)}\,\mathbf{n}\cdot\nabla[\cdot(\underline{\mathbf{I}}-\mathbf{nn})] \qquad [51]$$

$$\mathbf{v}\cdot\mathbf{n}=0 \qquad [52]$$

where $\mathbf{n}$ is a unit vector normal to the interface, pointing out of the porous medium, and $\mathbf{v}_{\text{int}}$ is the velocity of the interface. At the interface between porous and impermeable media we have:

$$(\hat{\mathbf{v}}-\mathbf{v}_{\text{int}})\cdot\mathbf{n}=-\sqrt{(k)}\,\mathbf{n}\cdot\nabla(\hat{\mathbf{v}}\cdot\mathbf{n}) \qquad [53]$$

where $\mathbf{n}$ is a unit vector normal to the interface, pointing into the porous medium.

An alternative form of [51] and [53] is, respectively:

$$\mathbf{v}_t-(\mathbf{v}_{\text{int}})_t=\sqrt{(k)}\,\mathbf{n}\cdot\nabla\mathbf{v}_t \qquad [54]$$

$$\hat{v}_n-(v_{\text{int}})_n=\sqrt{(k)}\,\nabla_t\cdot\hat{\mathbf{v}}_t \qquad [55]$$

where the suffices n and t denote components normal and tangential to the boundary respectively. Equation [55] is obtained by application of the continuity equation on the interface.

7. CONCLUSIONS

In the present work the problem of the viscous flow relative to an isolate permeable sphere with an impermeable core has been discussed. A complete analytical solution has been found and compared with the result obtained by applying Darcy's law.

The conclusions may be summarized as follows:

(a) For large values of the non-dimensional radius $r/\sqrt{(k)}$ (that is large particles or low permeability), the main difference between the two solutions is the presence of a boundary layer in the Brinkman result for the velocity field at the edges of the porous medium, which is incompatible with Darcy's law.

(b) Particular boundary conditions must be applied to Darcy's law in order to fit it into Brinkman's solution out of the boundary layers. Such boundary conditions [50]–[53] have been discussed in detail, and manifest that physical intuition is not sufficient to determine the correct boundary conditions for a first order approximation. This stems from the fact that Darcy's equation is invalid close to the interface, and modified boundary conditions should be applied to account for this. Application of the common boundary condition $v_r=0$ results in an error of order $\sqrt{(k)}/a$ for the velocity and pressure fields.

(c) The existence of an impermeable core inside a porous particle has almost no effect on the flow field exterior to the particle.

(d) Although boundary conditions [50]–[53] were derived from the comparison of Darcy's and Brinkman's solutions of a particular case (i.e. a porous sphere with a rigid core), we believe them to be valid in general for interfaces between porous–rigid and porous–liquid media. A rigorous proof of the foregoing statement is not yet available, but a simple explanation does exist. The boundary layer thickness was proven to be of order $\sqrt{k}$; however, the local curvature of the interface is an order of magnitude larger and is determined by a characteristic length, say a, of the particle. Since the proposed boundary conditions account for a correction of order $\sqrt{k}/a$, only local geometry parameters that affect the boundary layer are to be considered, i.e. pores size when small compared to a. The same reasoning exists when solving boundary layer problems on curved surfaces for high Reynolds numbers. A first order solution accounts for the downstream distance to obtain the thickness of the boundary layer, and the detailed shape of the surface is of no significance.

(e) Boundary condition [55] can physically be viewed as a two dimensional continuity equation with a source term. This form is equivalent to the result obtained in Ekman's boundary layer. (Greenspan 1968, p. 32) where a normal velocity component is created of order $\sqrt{E}$—Ekman's number (in our case $\sqrt{k}$) to compensate for the transverse mass flow in the boundary layer.

(f) The application of Brinkman's equation made it possible to obtain an exact solution for the porous sphere with a rigid core. However, its validity for media with low porosities is questionable. Nevertheless, since the correction to the normal boundary condition is significant only for media with large porosities its utilization for these cases seems to be justified. Experimental data is, however, needed to verify this prediction. We suggest that, when this comparison between theory and experiment is made, a numerical correction coefficient is added to [55], namely,

$$\hat{v}_n - (v_{\rm int})_n = C\sqrt{(k)}\nabla_t \cdot \hat{\mathbf{v}}_t.$$

The deviations of the numerical value of C from 1 will account for a difference in the flow pattern in porous media from that predicted by Brinkman.

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